Topological Spaces

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Ms. R. MAHESWARI
Assistant Professor
Department of Mathematics
Shrimati Indira Gandhi College
Trichy.

1.The Real Line (R) with the Standard Topology:

- **Topology:**

  The standard topology on the set of real numbers is generated by open intervals (a,b), where a and b are real numbers. This means that any open set in this topology can be expressed as a union of such open intervals.

  Example: The interval (−1,1) is an open set. The union of (0,0.5) and (0.7,1.2) is also an open set. Concepts like continuity of functions you learned in calculus are defined with respect to this topology.

2. Euclidean Space (R^n) with the Standard Topology:

- **Topology:** This generalizes the real line to higher dimensions. In R^n, the standard topology is generated by open balls (or open n-dimensional boxes). An open ball centered at a point x with radius r>0 is the set of all points y such that the Euclidean distance between x and y is less than r.
- **Example:** In R^2 (the plane), an open ball is an open disk. The set $\{(x,y)\in R^2 | x^2+y^2<1\}$ is an open set in the standard topology on R^2. This topology is fundamental to multivariable calculus and analysis.

3. Metric Spaces and the Metric Topology:

- **Topology:** If you have a metric $d(x,y)$ defined on a set X (a function that measures the "distance" between any two points and satisfies certain properties), you can define a topology on X called the metric topology. The open sets in this topology are unions of open balls $B(x,r)=\{y \in X \mid d(x,y)<r\}$, where $x \in X$ and $r>0$.
- **Example:** The real line R with the absolute value as the metric ($d(x,y)=|x-y|$) induces the standard topology. Similarly, Euclidean space R^n with the Euclidean distance metric induces the standard topology on R^n. Many important topological spaces arise from metrics.

4. Subspace Topology:

- **Topology:** If Y is a subset of a topological space X with topology T, the subspace topology on Y consists of all intersections of open sets in T with Y. That is, a subset $U \subseteq Y$ is open in the subspace topology if and only if there exists an open set $V \in T$ such that $U=V \cap Y$.
- **Example:** Consider the unit circle $S1=\{(x,y) \in R^2 \mid x^2+y^2=1\}$ as a subspace of R^2 with the standard topology. An open set in the subspace topology on S1 is the intersection of an open disk in R^2 with the circle. For instance, the part of the circle lying in the open half-plane $x>0$ is an open set in the subspace topology.

5. Product Topology:

- **Topology:** If X and Y are topological spaces with topologies T_X and T_Y respectively, the product topology on the Cartesian product X×Y is the topology generated by the basis of sets of the form U×V, where $U \in T_X$ and $V \in T_Y$. This can be generalized to the product of any collection of topological spaces.
- **Example:** The standard topology on R^2 is the product topology of R×R, where R has the standard topology. An open rectangle (a,b)×(c,d) is a basic open set in this product topology.

6. Quotient Topology:

- **Topology:** If X is a topological space and p:X→Y is a surjective map, the quotient topology on Y is the finest topology on Y such that the map p is continuous. A subset V⊆Y is open in the quotient topology if and only if its preimage p−1(V) is open in X.
- **Example:** Consider the closed interval [0,1] with the subspace topology from R. If we identify the endpoints 0 and 1, we can define a surjective map p:[0,1]→S1 (the unit circle). The quotient topology on S1 induced by this map makes it a familiar topological space.

Basis

Definition:

Let (X,T) be a topological space. A collection $B \subseteq T$ of open subsets of X is called a **basis** for the topology T if every open set $U \in T$ can be expressed as the union of some subfamily of B.

In simpler terms, you can build any open set in the topology by taking appropriate unions of the sets in the basis. The sets in the basis are often called **basic open sets**.

A collection B of subsets of a set X is a basis for some topology on X if and only if it satisfies the following two conditions:

1. **Coverage:** The union of all the sets in B must be equal to X. [1] That is, every point in X must belong to at least one set in B:
2. **Intersection Property:** For every point $x \in X$ that lies in the intersection of two basic open sets $B_1, B_2 \in B$, there must exist another basic open set $B_3 \in B$ such that $x \in B_3$ and $B_3 \subseteq B_1 \cap B_2$.

If a collection B satisfies these two conditions, then the topology T generated by B consists of all possible unions of elements from B.

Connected space

Definition:

A topological space (X,T) is said to be **connected** if it cannot be expressed as the union of two non-empty, disjoint open subsets.

In other words, if $X=U\cup V$, where U and V are open in T, and $U\cap V=\emptyset$, then either $U=\emptyset$ or $V=\emptyset$ (or both, if X itself is empty).

A space which is not connected is called as disconnected space.

Examples of Connected Spaces:

- **The Real Line (R) with the Standard Topology:** It cannot be split into two non-empty disjoint open intervals (or unions of them).
- **Any Interval in R (open, closed, or half-open):** For example, (0,1), [a,b], $(-\infty,a]$, etc., are all connected with the subspace topology inherited from R.
- **Euclidean Space (Rn) with the Standard Topology:** For any $n\geq 1$, Rn is connected.

- **Continuous Images of Connected Spaces:** If $f:X\to Y$ is a continuous function and X is connected, then the image f(X) is also connected (with the subspace topology inherited from Y). This is a powerful way to show that many spaces are connected. For example, the circle S1 is the continuous image of the connected interval [0,1] under the map $t\mapsto(\cos(2\pi t),\sin(2\pi t))$, so S1 is connected.
- **The Union of Connected Sets with a Non-Empty Intersection:** If {Aα}α∈I is a collection of connected subsets of a topological space X such that ⋂α∈IAα偃=∅, then their union ⋃α∈IAα is also connected.

Examples of Disconnected Spaces:

- **The union of two disjoint open intervals in R:** For example, $(0,1)\cup(2,3)$ is disconnected because (0,1) and (2,3) are non-empty, disjoint, and open in the subspace topology.
- **A discrete space with more than one point:** In a discrete space, every subset is open (and closed). If the space has at least two points, say {x,y}, then {x} and {y} are non-empty, disjoint open sets whose union is the entire space.
- **The rational numbers (Q) with the subspace topology from R:** For any two rationals $p<q$, we can find an irrational number r such that $p<r<q$. Then $Q=(Q\cap(-\infty,r))\cup(Q\cap(r,\infty))$, which are non-empty, disjoint, and open in the subspace topology on Q.

Connected Components:

Every topological space can be uniquely decomposed into maximal connected subspaces called its **connected components**.

- The connected component of a point x in a space X is the union of all connected subsets of X that contain x. This union is itself connected and is the largest connected subset containing x.
- The connected components of a topological space form a partition of the space: they are disjoint, non-empty, and their union is the entire space.
- Connected components are always closed sets.

Definition:

A topological space (X,T) is said to be **compact** if every open cover of X has a finite subcover.

- **Open Cover:** An open cover of a space X is a collection of open sets $\{U\alpha\}\alpha\in I$ (where I is an index set, possibly infinite) such that their union contains X: $X\subseteq_{\alpha\in I}\bigcup U\alpha$
- **Finite Subcover:** A finite subcover is a finite subcollection $\{U\alpha_1,U\alpha_2,\ldots,U\alpha_n\}$ of the original open cover such that their union still contains X: $X\subseteq i=1\bigcup_n U\alpha_i$

So, a space is compact if, whenever you cover it with any collection of open sets, you can always find a finite number of those open sets that still cover the entire space.

Key Properties of Compact Spaces:

- **Closed Subsets of Compact Spaces are Compact:** If Y is a closed subset of a compact space X, then Y (with the subspace topology) is also compact.
- **Compact Subspaces of Hausdorff Spaces are Closed:** If Y is a compact subspace of a Hausdorff space X, then Y is a closed subset of X. (A Hausdorff space is one where any two distinct points have disjoint open neighborhoods.)
- **Continuous Images of Compact Spaces are Compact:** If $f: X \to Y$ is a continuous function and X is a compact space, then the image f(X) (with the subspace topology inherited from Y) is also compact. This is a very powerful property for proving compactness of new spaces.
- **Compactness in Metric Spaces is Equivalent to Sequential Compactness:** For a metric space, compactness is equivalent to the property that every sequence in the space has a subsequence that converges to a point within the space. It is also equivalent to being complete and totally bounded.
- **The Product of Finitely Many Compact Spaces is Compact (Tychonoff's Theorem extends this to arbitrary products):** If $X1, X2, \ldots, Xn$ are compact spaces, then their product $X1 \times X2 \times \ldots \times Xn$ (with the product topology) is also compact.

Examples of Compact Spaces:

- **Any Finite Topological Space:** Since any cover will be finite, it automatically has a finite subcover (itself).
- **A Set with the Cofinite Topology:** In this topology, a subset is open if and only if it is empty or its complement is finite. Any open cover of such a space will have at least one non-empty open set U. The complement of U is finite, so you can pick one open set from the cover to contain each of these finitely many points, resulting in a finite subcover.
- **The Closed Unit Interval [0,1] in R (with the standard topology):** This is a fundamental example, often proven using the Heine-Borel Theorem.
- **Any Closed and Bounded Interval [a,b] in R.**
- **Closed and Bounded Subsets of Rn (with the standard topology) - (Heine-Borel Theorem):** For example, closed disks, closed balls, closed rectangles, etc.
- **The Unit Circle S1 and the n-Sphere Sn (with the subspace topology from Rn+1):** These are continuous images of compact intervals.
- **The Cantor Set:** A fascinating example that is compact, totally disconnected, and perfect.

Examples of Non-Compact Spaces:

- **The Open Interval (0,1) in R:** The open cover $\{(1/n,1) \mid n \in N, n>1\}$ has no finite subcover.
- **The Real Line R itself:** The open cover $\{(n-1,n+1) \mid n \in Z\}$ has no finite subcover.
- **Any Infinite Set with the Discrete Topology:** The open cover consisting of all singleton sets $\{\{x\}\}x \in X$ has no finite subcover.
- **The Rational Numbers Q (with the subspace topology from R):** Even though it's bounded within a compact interval, it's not closed in R, and it's not compact.

Thank You